卡尔曼滤波（状态估计）

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The origin of the Kalman Filtering

卡尔曼全名Rudolf Emil Kalman，匈牙利数学家，1930年出生于匈牙利首都布达佩斯。1953，1954年于麻省理工学院分别获得电机工程学士及硕士学位。1957年于哥伦比亚大学获得博士学位。
他的博士论文和1960年发表的论文《A New Approach to Linear Filtering and Prediction Problems》（线性滤波与预测问题的新方法）

Example(math)

Consider the following simplest problem.
Noisy measurements,y1,y2,...,yk. an unknown constant x

$$y_k = x + v_k \tag{1}$$

and v_k,$k = 1, 2, ...$(independent and identically distributed noises)
A simple common sense

$$\hat{x_k} = \frac{1}{k}\sum_{i=1}^{k} y_i \cdots\cdots estimate\ of\ x\ after\ k\ measurements \tag{2}$$

is an optimal estimate that converges to the true x as $k \to \infty$

Now we can rewrite Eq.(2) as

$$\hat{x}_k = \frac{1}{k}\sum_{i=1}^{k-1} y_i + \frac{1}{k} y_k = \frac{k-1}{k}\frac{1}{k-1}\sum_{i=1}^{k-1} + \frac{1}{k} y_k = \hat{x}_{k-1} + \frac{1}{k}(y_k - \hat{x}_{k-1}) \tag{3}$$

The best estimate of x after k measurements is the best estimate of x after k-1 measurements plus a correction term $y_k - \hat{x}_{k-1}$

The weighting factor is $\frac{1}{k}$. initially, we don't think our estimates are good when k is small. We must pay more attention to the correction term. As k becomes large, so does the confidence in our estimate. Thus we pay less attention to the correction term.

Label the weighting factor $\frac{1}{k}$ as P_k, the recursive equation

$$P_k = P_{k-1} - P_{k-1}(P_{k-1} + 1)^{-1} P_{k-1} \qquad (Ho\ 1963) \tag{4}$$

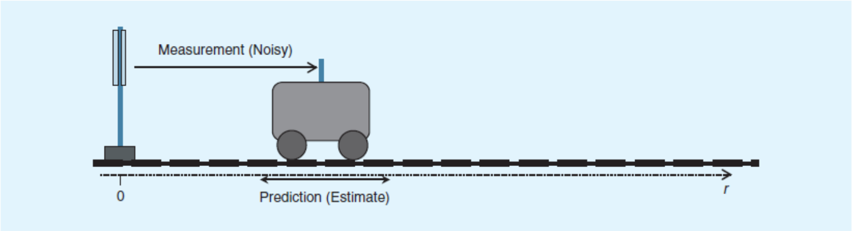
Example (physic)

A train(or car) is moving along a railway line.

The best possible estimate of the location of the train. Information is available from two sources:

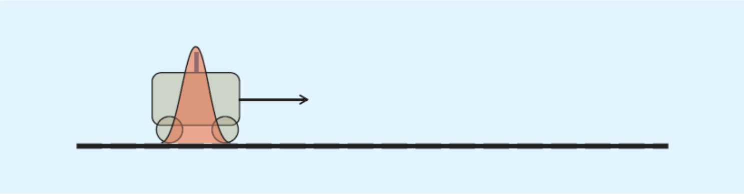
1) predictions based on the last known position and velocity of the train

2) measurements from a radio ranging system deployed at the track side.



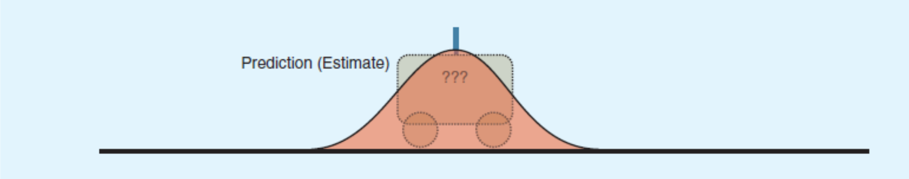
[FIG1] This figure shows the one-dimensional system under consideration.

The initial state of the system($t = 0s$) is known to a reasonable accuracy, as shown in Figure 2.



[FIG2] The initial knowledge of the system at time $t = 0$. The red Gaussian distribution represents the pdf providing the initial confidence in the estimate of the position of the train. The arrow pointing to the right represents the known initial velocity of the train.

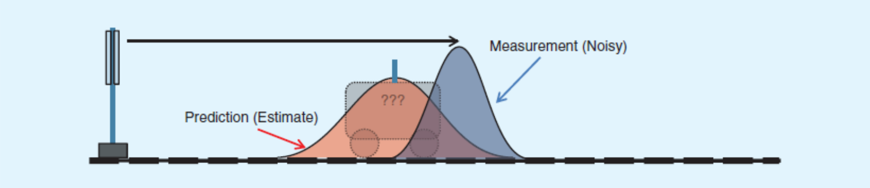
A prediction of the new position of the train($t = 1s$)



[FIG3] Here, the prediction of the location of the train at time $t = 1$ and the level of uncertainty in that prediction is show
confidence in the knowledge of the position of the train has decreased, as we are not certain if the train has undergone
accelerations or decelerations in the intervening period from $t = 0$ to $t = 1$.

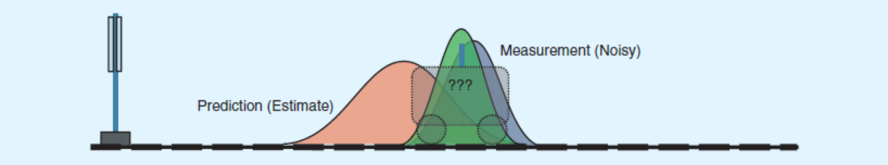
The accuracy of out position estimate compared to $t = 0s$, due to the uncertainty associated with any process noise from accelerations or decelerations undertaken from $t = 0$ to $t = 1$.

At $t = 1$, we make a measurement of the location of the train using the radio positioning system(无线电定位系统), and this is represented by the blue Gaussian pdf in Figure 4.



[FIG4] Shows the measurement of the location of the train at time $t = 1$ and the level of uncertainty in that noisy measurement, represented by the blue Gaussian pdf. The combined knowledge of this system is provided by multiplying these two pdfs together.

The best estimate we can make of the location of the train is provided by combining our knowledge from the prediction and the measurement. This is achieved by multiplying the two corresponding pdfs together.



[FIG5] Shows the new pdf (green) generated by multiplying the pdfs associated with the prediction and measurement of the train's location at time $t = 1$. This new pdf provides the best estimate of the location of the train, by fusing the data from the prediction and the measurement.

The product of two Gaussian functions is another Gaussian function. This is the key to the elegant recursive properties of the Kalman filter.

mathematically

The red Gaussian function(prediction)

$$y_1(r;\mu_1,\sigma_1)=\frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{(r-\mu_1)^2}{2\sigma_1^2}} \tag{5}$$

The blue Gaussian function(measurement)

$$y_2(r;\mu_2,\sigma_2)=\frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{(r-\mu_2)^2}{2\sigma_2^2}} \tag{6}$$

The green Gaussian function(measurement and prediction)

$$\begin{aligned} y_{fused}(r;\mu_1,\sigma_1,\mu_2,\sigma_2) &= \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{(r-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{(r-\mu_2)^2}{2\sigma_2^2}} \\ &= \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2}}e^{-\left(\frac{(r-\mu_1)^2}{2\sigma_1^2}+\frac{(r-\mu_2)^2}{2\sigma_2^2}\right)} \end{aligned} \tag{7}$$

$$y_{fused}(r;\mu_{fused},\sigma_{fused}) = \frac{1}{\sqrt{2\pi\sigma_{fused}^2}} e^{-\frac{(r-\mu_{fused})^2}{2\sigma_{fused}^2}} \tag{8}$$

where

$$\mu_{fused} = \frac{\mu_1\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} = \mu_1 + \frac{\sigma_1^2(\mu_2-\mu_1)}{\sigma_1^2+\sigma_2^2}$$

and

$$\sigma_{fused}^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \sigma_1^2 - \frac{\sigma_1^4}{\sigma_1^2+\sigma_2^2}$$

substituting $K = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$

$\mu_{fused} = \mu_1 + K(\mu_2 - \mu_1)$ and $\sigma_{fused}^2 = \sigma_1^2 - K\sigma_1^2$

K,the kalman gain 估计量方差占总方差的比重

五条公式

$$\mathbf{X}(k) = \mathbf{\Phi}(k, k-1) \cdot \mathbf{X}(k-1) + \mathbf{\Gamma}(k, k-1) \cdot \omega(k-1) \tag{9}$$

$$\mathbf{y}(k) = \mathbf{h}(k) \cdot \mathbf{X}(k) + \nu(k) \tag{10}$$

Kalman Filtering

$$\boxed{\hat{x}(k|k-1) = \Phi(k, k-1)\hat{x}(k-1|k-1)}$$

$$\boxed{P_{xx}(k|k-1) = \Phi(k, k-1)\hat{p}(k-1|k-1)\Phi^T(k, k-1)}$$

$$+\,\Gamma(k, k-1)Q_\omega(k-1)\Gamma^T(k, k-1)$$

$$\boxed{K(k) = P_{xy}(k|k-1)P_{yy}^{-1}(k|k-1)}$$

$$\boxed{\hat{x}(k|k) = \hat{x}(k|k-1) + K(k)[y(k) - \hat{y}(k|k-1)]}$$

$$\boxed{\hat{p}(k|k) = [I - K(k)h(k)]P_{xx}(k|k-1)}$$