The Reynolds Analogy Factor: Laminar Boundary Layer, and What Changes When It Is Turbulent

1 Problem, definitions and governing equations

Setting

A uniform stream of velocity U_∞ and temperature Θ_∞ flows over a semi-infinite flat plate at zero incidence, held at temperature Θ_w. The coordinate x is the streamwise development length measured from the leading edge, and z is the wall-normal direction. The corresponding velocity components are u (streamwise) and w (wall-normal). We assume

1. steady, two-dimensional flow, laminar unless stated otherwise;
2. constant properties ρ, μ, k, c_p, with $\nu = \mu/\rho$, $\alpha = k/(\rho c_p)$ and $\mathrm{Pr} = \nu/\alpha$;
3. zero pressure gradient, $dp_\infty/dx = 0$;
4. negligible viscous dissipation and buoyancy;
5. $Re_x = U_\infty x/\nu \gg 1$ and $Pe_x = U_\infty x/\alpha = Re_x \mathrm{Pr} \gg 1$.

Definitions

With $\Delta\Theta \equiv \Theta_\mathrm{w} - \Theta_\infty$,

$$\tau_\mathrm{w} = \mu \left.\frac{\partial u}{\partial z}\right|_{z=0}, \qquad q''_\mathrm{w} = -k \left.\frac{\partial \Theta}{\partial z}\right|_{z=0}, \tag{1}$$

$$\frac{C_{f,x}}{2} = \frac{\tau_\mathrm{w}}{\rho U_\infty^2}, \qquad St_x = \frac{q''_\mathrm{w}}{\rho c_p U_\infty \, \Delta\Theta} = \frac{Nu_x}{Re_x \, \mathrm{Pr}}, \qquad Nu_x = \frac{q''_\mathrm{w}\, x}{k \, \Delta\Theta}. \tag{2}$$

The Reynolds analogy factor is therefore

$$\boxed{\mathcal{R} \equiv \frac{St_x}{C_{f,x}/2} = \frac{q''_\mathrm{w}}{\Delta\Theta} \, \frac{U_\infty}{c_p \, \tau_\mathrm{w}}} \tag{3}$$

Full equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \tag{5}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right), \tag{6}$$

$$u\frac{\partial \Theta}{\partial x} + w\frac{\partial \Theta}{\partial z} = \alpha\left(\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial z^2}\right), \tag{7}$$

with $u = w = 0$, $\Theta = \Theta_\mathrm{w}$ at $z = 0$ ($x > 0$), and $u \to U_\infty$, $\Theta \to \Theta_\infty$ as $z \to \infty$.

2 Laminar boundary layer: inertia balances molecular diffusion

This section contains the central physical result for the laminar case. Across the whole boundary layer, convection (inertia) is balanced by *molecular* diffusion. That single balance fixes the momentum and thermal layer thicknesses, and through them the wall fluxes.

Order-of-magnitude analysis of the momentum equation

Let $\delta(x)$ be the thickness over which u changes from 0 to U_∞. Inside this layer

$$u \sim U_\infty, \qquad \frac{\partial}{\partial x} \sim \frac{1}{x}, \qquad \frac{\partial}{\partial z} \sim \frac{1}{\delta}, \qquad \delta \ll x. \tag{8}$$

Step 1: the scale of w (continuity). Both terms in (4) must be of the same order, so

$$\frac{U_\infty}{x} \sim \frac{w}{\delta} \quad \Longrightarrow \quad w \sim U_\infty \frac{\delta}{x} \ll U_\infty. \tag{9}$$

Step 2: the inertia terms.

$$u\frac{\partial u}{\partial x} \sim U_\infty \cdot \frac{U_\infty}{x} = \frac{U_\infty^2}{x}, \qquad w\frac{\partial u}{\partial z} \sim U_\infty \frac{\delta}{x} \cdot \frac{U_\infty}{\delta} = \frac{U_\infty^2}{x}. \tag{10}$$

Both convective terms are of order U_∞^2/x.

Step 3: the viscous terms.

$$\nu\frac{\partial^2 u}{\partial z^2} \sim \nu\frac{U_\infty}{\delta^2}, \qquad \nu\frac{\partial^2 u}{\partial x^2} \sim \nu\frac{U_\infty}{x^2} = \left(\frac{\delta}{x}\right)^2 \nu\frac{U_\infty}{\delta^2} \ll \nu\frac{U_\infty}{\delta^2}. \tag{11}$$

Streamwise diffusion is smaller by $(\delta/x)^2$ and is dropped.

Step 4: pressure. Every term of (6) is smaller than the corresponding term of (5) by a factor δ/x, so $\partial p/\partial z \sim \rho U_\infty^2 \delta/x^2$. The pressure change across the layer, $\Delta p \sim \rho U_\infty^2 (\delta/x)^2$, is negligible; hence $p \approx p_\infty(x)$ and $\partial p/\partial x = 0$.

Step 5: the balance. The boundary layer is the region in which viscous diffusion is as large as inertia:

$$\boxed{\frac{U_\infty^2}{x} \sim \nu\frac{U_\infty}{\delta^2}} \quad \Longrightarrow \quad \delta^2 \sim \frac{\nu x}{U_\infty} \quad \Longrightarrow \quad \frac{\delta}{x} \sim Re_x^{-1/2}, \qquad \frac{C_{f,x}}{2} \sim \frac{\nu}{U_\infty \delta} \sim Re_x^{-1/2}. \tag{12}$$

In the time x/U_∞ that fluid takes to travel a distance x, momentum diffuses a distance $\sqrt{\nu x/U_\infty}$. The assumption $\delta \ll x$ is self-consistent if and only if $Re_x \gg 1$. The same argument applied to (7) drops $\alpha\,\partial^2\Theta/\partial x^2$ provided $\delta_\theta \ll x$, which requires $Pe_x \gg 1$.

Boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{13}$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2}, \tag{14}$$

$$u\frac{\partial \Theta}{\partial x} + w\frac{\partial \Theta}{\partial z} = \alpha\frac{\partial^2 \Theta}{\partial z^2}. \tag{15}$$

The analogy factor as a ratio of two thicknesses

Define two thicknesses from the wall gradients:

$$\delta_\tau \equiv \frac{U_\infty}{(\partial u/\partial z)_\mathrm{w}} = \frac{\mu U_\infty}{\tau_\mathrm{w}}, \qquad \delta_q \equiv \frac{\Delta\Theta}{-(\partial\Theta/\partial z)_\mathrm{w}} = \frac{k\,\Delta\Theta}{q''_\mathrm{w}} = \frac{x}{Nu_x}. \tag{16}$$

Substituting $\tau_\mathrm{w} = \mu U_\infty/\delta_\tau$ and $q''_\mathrm{w} = k\Delta\Theta/\delta_q$ into (3) gives the exact identity

$$\boxed{\mathcal{R} = \frac{1}{\mathrm{Pr}}\,\frac{\delta_\tau}{\delta_q}} \tag{17}$$

so the problem reduces to finding δ_θ/δ as a function of Pr.

Pr = 1: the exact Reynolds analogy

With $\phi = (\Theta - \Theta_\mathrm{w})/(\Theta_\infty - \Theta_\mathrm{w})$ and $\hat{u} = u/U_\infty$, equations (14) and (15) are

$$u\,\partial_x\hat{u} + w\,\partial_z\hat{u} = \nu\,\partial_z^2\hat{u}, \qquad u\,\partial_x\phi + w\,\partial_z\phi = \alpha\,\partial_z^2\phi, \tag{18}$$

with identical boundary conditions. If $\nu = \alpha$, then $\phi \equiv \hat{u}$ and $St_x = C_{f,x}/2$ exactly. A pressure gradient would appear only in the momentum equation and break this equivalence.

Pr ≫ 1: thin thermal layer, $\delta_\theta \ll \delta$

The thermal layer lies inside the linear part of the velocity profile:

$$u \approx \frac{\tau_\mathrm{w}}{\mu}\,z \sim U_\infty\frac{z}{\delta} \quad\Longrightarrow\quad u\big|_{z\sim\delta_\theta} \sim U_\infty\frac{\delta_\theta}{\delta}. \tag{19}$$

Since $\delta \propto x^{1/2}$, $\partial u/\partial x \sim U_\infty z/(\delta x)$, and continuity gives $w = -\int_0^z \partial_x u\,dz' \sim U_\infty\delta_\theta^2/(\delta x)$ inside the thermal layer. Both convection terms are then of the same order:

$$u\frac{\partial\Theta}{\partial x} \sim w\frac{\partial\Theta}{\partial z} \sim U_\infty\frac{\delta_\theta}{\delta}\frac{\Delta\Theta}{x}. \tag{20}$$

Balancing with conduction, $\alpha\Delta\Theta/\delta_\theta^2$:

$$U_\infty\frac{\delta_\theta}{\delta\,x} \sim \frac{\alpha}{\delta_\theta^2} \implies \left(\frac{\delta_\theta}{\delta}\right)^3 \sim \frac{\alpha x}{U_\infty\delta^2} \sim \frac{\alpha}{\nu} \implies \boxed{\frac{\delta_\theta}{\delta} \sim \mathrm{Pr}^{-1/3}, \qquad \mathcal{R} \sim \mathrm{Pr}^{-1}\mathrm{Pr}^{1/3} = \mathrm{Pr}^{-2/3}} \tag{21}$$

which is self-consistent for $\mathrm{Pr} \gg 1$. Also $\delta_\theta/x \sim Re_x^{-1/2}\mathrm{Pr}^{-1/3}$ and $Nu_x \sim Re_x^{1/2}\mathrm{Pr}^{1/3}$.

Pr ≪ 1: thick thermal layer, $\delta_\theta \gg \delta$

Most of the thermal layer lies outside the velocity layer, where $u \approx U_\infty$ (slug flow) and w tends to the displacement velocity $w_\infty \sim U_\infty\delta/x$. The convection terms are

$$u\frac{\partial\Theta}{\partial x} \sim U_\infty\frac{\Delta\Theta}{x}, \qquad w\frac{\partial\Theta}{\partial z} \sim \frac{\delta}{\delta_\theta}\,U_\infty\frac{\Delta\Theta}{x}, \tag{22}$$

so $w\,\partial_z\Theta$ is smaller by δ/δ_θ and is dropped at leading order. Balancing with conduction:

$$U_\infty\frac{\Delta\Theta}{x} \sim \alpha\frac{\Delta\Theta}{\delta_\theta^2} \implies \frac{\delta_\theta}{x} \sim Pe_x^{-1/2} \implies \boxed{\frac{\delta_\theta}{\delta} \sim \mathrm{Pr}^{-1/2}, \qquad \mathcal{R} \sim \mathrm{Pr}^{-1}\mathrm{Pr}^{1/2} = \mathrm{Pr}^{-1/2}} \tag{23}$$

which is self-consistent for $\mathrm{Pr} \ll 1$, with $Nu_x \sim Pe_x^{1/2}$. Neglecting $\alpha\,\partial^2\Theta/\partial x^2$ requires $Pe_x \gg 1$; for $\mathrm{Pr} \ll 1$ this leading-edge restriction is $1/\mathrm{Pr}$ times longer than the momentum one.

Key result for the laminar case

	velocity seen by thermal layer	δ_θ/δ	$\mathcal{R} = St_x/(C_{f,x}/2)$
$\mathrm{Pr} \gg 1$	$U_\infty\,\delta_\theta/\delta$ (linear shear)	$\mathrm{Pr}^{-1/3}$	$\mathrm{Pr}^{-2/3}$
$\mathrm{Pr} = 1$	identical profiles	1	1 (exact)
$\mathrm{Pr} \ll 1$	U_∞ (slug flow)	$\mathrm{Pr}^{-1/2}$	$\mathrm{Pr}^{-1/2}$

Every entry follows from one statement: *convection balances molecular diffusion over the entire layer*, for momentum ($U_\infty^2/x \sim \nu U_\infty/\delta^2$) and for heat ($u\,\Delta\Theta/x \sim \alpha\,\Delta\Theta/\delta_\theta^2$). The $\mathrm{Pr}^{-2/3}$ law is the large-Pr limit; for $\mathrm{Pr} \ll 1$ the law is $\mathrm{Pr}^{-1/2}$. Because δ and δ_θ both scale as $x^{1/2}$, $\mathcal{R}$ depends on Pr only, not on Re_x.

3 Laminar boundary layer: exact solution and asymptotic limits

Momentum: the Blasius equation

The scaling (12) suggests

$$\eta = z\sqrt{\frac{U_\infty}{\nu x}}, \qquad \psi = \sqrt{\nu U_\infty x}\, f(\eta), \qquad u = \frac{\partial\psi}{\partial z}, \quad w = -\frac{\partial\psi}{\partial x}. \tag{24}$$

With $\partial\eta/\partial z = \sqrt{U_\infty/\nu x}$ and $\partial\eta/\partial x = -\eta/(2x)$,

$$u = U_\infty f'(\eta), \qquad w = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\,\big(\eta f' - f\big), \tag{25}$$

$$\frac{\partial u}{\partial x} = -\frac{U_\infty}{2x}\eta f'', \qquad \frac{\partial u}{\partial z} = U_\infty\sqrt{\frac{U_\infty}{\nu x}}\, f'', \qquad \frac{\partial^2 u}{\partial z^2} = \frac{U_\infty^2}{\nu x} f'''. \tag{26}$$

Substituting into (14), the $\eta f' f''$ terms cancel and $-\frac{U_\infty^2}{2x} f f'' = \frac{U_\infty^2}{x} f'''$:

$$\boxed{f''' + \tfrac{1}{2} f f'' = 0, \qquad f(0) = f'(0) = 0, \quad f'(\infty) = 1.} \tag{27}$$

Numerically, $f''(0) \equiv a = 0.332057$ and $f \to \eta - \beta$ with $\beta = 1.72079$. Hence

$$\frac{C_{f,x}}{2} = a\, Re_x^{-1/2} = 0.332\, Re_x^{-1/2}, \qquad \frac{\delta_{99}}{x} = 4.910\, Re_x^{-1/2}, \qquad \frac{\delta_\tau}{x} = 3.012\, Re_x^{-1/2}, \tag{28}$$

$$\frac{\delta^*}{x} = \beta\, Re_x^{-1/2} = 1.721\, Re_x^{-1/2}, \qquad \frac{w_\infty}{U_\infty} = \frac{\beta}{2} Re_x^{-1/2} = 0.860\, Re_x^{-1/2}. \tag{29}$$

Energy equation in similarity form

Let $\theta(\eta) = (\Theta - \Theta_\mathrm{w})/(\Theta_\infty - \Theta_\mathrm{w})$, so $\theta(0) = 0$ and $\theta(\infty) = 1$. Then

$$\frac{\partial\Theta}{\partial x} = (\Theta_\infty - \Theta_\mathrm{w})\Big(-\frac{\eta}{2x}\Big)\theta', \quad \frac{\partial\Theta}{\partial z} = (\Theta_\infty - \Theta_\mathrm{w})\sqrt{\frac{U_\infty}{\nu x}}\,\theta', \quad \frac{\partial^2\Theta}{\partial z^2} = (\Theta_\infty - \Theta_\mathrm{w})\frac{U_\infty}{\nu x}\theta''. \tag{30}$$

Inserting these and (25) into (15):

$$U_\infty f'\Big(-\frac{\eta}{2x}\theta'\Big) + \frac{U_\infty}{2x}(\eta f' - f)\theta' = \alpha\frac{U_\infty}{\nu x}\theta'' \quad\Longrightarrow\quad \boxed{\theta'' + \frac{\mathrm{Pr}}{2} f\,\theta' = 0.} \tag{31}$$

Closed-form solution

Equation (31) is first order in θ', so

$$\theta'(\eta) = \theta'(0)\exp\Big[-\frac{\mathrm{Pr}}{2}\int_0^\eta f\,d\eta'\Big], \qquad \theta'(0) = \frac{1}{I(\mathrm{Pr})}, \qquad I(\mathrm{Pr}) = \int_0^\infty \exp\Big[-\frac{\mathrm{Pr}}{2}\int_0^\eta f\,d\eta'\Big]d\eta. \tag{32}$$

The Blasius equation gives $f'''/f'' = -\frac{1}{2}f$, i.e. $f''(\eta) = a\exp[-\frac{1}{2}\int_0^\eta f]$, so

$$I(\mathrm{Pr}) = \int_0^\infty \left[\frac{f''(\eta)}{f''(0)}\right]^{\mathrm{Pr}} d\eta. \tag{33}$$

At $\mathrm{Pr} = 1$, $I = 1/a$ and $\theta = f'$.

Heat transfer and the exact analogy factor

$$q''_{\mathrm{w}} = k(\Theta_{\mathrm{w}} - \Theta_\infty)\sqrt{\frac{U_\infty}{\nu x}}\,\theta'(0) \implies Nu_x = \theta'(0)\,Re_x^{1/2}, \qquad St_x = \frac{\theta'(0)}{\mathrm{Pr}}Re_x^{-1/2}. \tag{34}$$

Dividing by $C_{f,x}/2 = a\,Re_x^{-1/2}$, the Reynolds number cancels:

$$\boxed{\mathcal{R}(\mathrm{Pr}) = \frac{\theta'(0)}{\mathrm{Pr}\,a} = \frac{1}{\mathrm{Pr}\displaystyle\int_0^\infty \big(f''(\eta)\big)^{\mathrm{Pr}}\,a^{1-\mathrm{Pr}}\,d\eta}} \tag{35}$$

Large-Pr limit: the origin of $\mathrm{Pr}^{-2/3}$

For $\mathrm{Pr} \gg 1$ the integrand of I decays at small η, where

$$f = \frac{a}{2}\eta^2 - \frac{a^2}{240}\eta^5 + \ldots, \qquad \int_0^\eta f\,d\eta' = \frac{a}{6}\eta^3\Big(1 - \frac{a}{240}\eta^3 + \ldots\Big), \tag{36}$$

using $f'''(0) = f^{(4)}(0) = 0$ and $f^{(5)}(0) = -a^2/2$. With $s = (\mathrm{Pr}\,a/12)^{1/3}\eta$,

$$I = \Big(\frac{12}{\mathrm{Pr}\,a}\Big)^{1/3}\int_0^\infty e^{-s^3}ds\,\big[1 + O(\mathrm{Pr}^{-1})\big] = \Big(\frac{12}{\mathrm{Pr}\,a}\Big)^{1/3}\Gamma\big(\tfrac{4}{3}\big)\,\big[1 + O(\mathrm{Pr}^{-1})\big], \tag{37}$$

$$Nu_x = 0.3387\,Re_x^{1/2}\mathrm{Pr}^{1/3}, \qquad \boxed{\mathcal{R} \to 1.0200\,\mathrm{Pr}^{-2/3}\big[1 + O(\mathrm{Pr}^{-1})\big]} \tag{38}$$

The profile is $\theta = \int_0^s e^{-t^3}dt/\Gamma(4/3)$, which reaches 0.99 at $s = 1.4037$:

$$\frac{\delta_{\theta,99}}{x} = 4.641\,Re_x^{-1/2}\mathrm{Pr}^{-1/3}, \qquad \frac{\delta_{\theta,99}}{\delta_{99}} = 0.945\,\mathrm{Pr}^{-1/3}, \qquad \frac{\delta_q}{x} = 2.952\,Re_x^{-1/2}\mathrm{Pr}^{-1/3}. \tag{39}$$

Small-Pr limit: slug flow and $\mathrm{Pr}^{-1/2}$

For $\mathrm{Pr} \ll 1$ the integrand decays at $\eta = O(\mathrm{Pr}^{-1/2})$. Write exactly

$$\int_0^\eta f\,d\eta' = \frac{\eta^2}{2} - \beta\eta + c(\eta), \qquad c(\eta) = \int_0^\eta \big(f - \eta' + \beta\big)d\eta', \tag{40}$$

where $c(\eta)$ is bounded, so $e^{-\mathrm{Pr}\,c/2} = 1 + O(\mathrm{Pr})$ uniformly. With $s = \sqrt{\mathrm{Pr}}\,\eta/2$,

$$I = \frac{2}{\sqrt{\mathrm{Pr}}}\int_0^\infty e^{-s^2}\big(1 + \beta\sqrt{\mathrm{Pr}}\,s + O(\mathrm{Pr})\big)ds = \sqrt{\frac{\pi}{\mathrm{Pr}}}\Big(1 + \beta\sqrt{\frac{\mathrm{Pr}}{\pi}} + O(\mathrm{Pr})\Big), \tag{41}$$

$$Nu_x = \frac{1}{\sqrt{\pi}} Pe_x^{1/2}\big(1 - 0.971\,\mathrm{Pr}^{1/2}\big), \qquad \boxed{\mathcal{R} \to 1.6991\,\mathrm{Pr}^{-1/2}\big(1 - 0.971\,\mathrm{Pr}^{1/2} + O(\mathrm{Pr})\big)} \quad (42)$$

The correction is the displacement effect: β measures the Blasius velocity deficit, i.e. the outward velocity w_∞, and its relative size $\delta/\delta_\theta \sim \mathrm{Pr}^{1/2}$ matches the scaling of Section 2. At leading order $\theta = \mathrm{erf}\big(z/(2\sqrt{\alpha x/U_\infty})\big)$, and since $\mathrm{erf}(1.8214) = 0.99$,

$$\frac{\delta_{\theta,99}}{x} = 3.643\,Pe_x^{-1/2}, \qquad \frac{\delta_{\theta,99}}{\delta_{99}} = 0.742\,\mathrm{Pr}^{-1/2}, \qquad \frac{\delta_q}{x} = \sqrt{\pi}\,Pe_x^{-1/2} = 1.772\,Pe_x^{-1/2}. \quad (43)$$

Integral-method cross-check

With cubic profiles $u/U_\infty = \frac{3}{2}\zeta - \frac{1}{2}\zeta^3$ ($\zeta = z/\delta$) and $\theta = \frac{3}{2}\zeta_t - \frac{1}{2}\zeta_t^3$ ($\zeta_t = z/\delta_\theta$), the momentum integral gives $\delta/x = 4.641\,Re_x^{-1/2}$ and $C_{f,x}/2 = 0.3232\,Re_x^{-1/2}$ (exact 0.3321). The energy integral gives

- for $r = \delta_\theta/\delta \le 1$: $r^3 = 13/(14\,\mathrm{Pr})$, so $r = 0.976\,\mathrm{Pr}^{-1/3}$, $Nu_x = 0.331\,Re_x^{1/2}\mathrm{Pr}^{1/3}$ and $\mathcal{R} = 1/(r\,\mathrm{Pr}) = 1.025\,\mathrm{Pr}^{-2/3}$;
- for slug flow: $\delta_\theta^2 = 8\alpha x/U_\infty$, so $Nu_x = 0.530\,Pe_x^{1/2}$ (exact 0.564) and $\mathcal{R} = 1.641\,\mathrm{Pr}^{-1/2}$ (exact 1.699).

Numerical results for all Pr

Table 1 gives the exact similarity results. The column n is the local exponent $n = -d\ln\mathcal{R}/d\ln\mathrm{Pr}$.

Table 1: Exact laminar similarity solution, isothermal wall.

Pr	$Nu_x/Re_x^{1/2}$	$\mathcal{R}$	$\mathcal{R}/\mathrm{Pr}^{-2/3}$	$\mathcal{R}/\mathrm{Pr}^{-1/2}$	n	$\delta_{\theta,99}/\delta_{99}$
0.001	0.0173	52.15	0.522	1.649	0.515	23.72
0.01	0.0516	15.54	0.721	1.554	0.541	7.677
0.03	0.0844	8.472	0.818	1.467	0.564	4.543
0.1	0.1400	4.217	0.909	1.334	0.596	2.608
0.3	0.2148	2.156	0.966	1.181	0.625	1.618
0.7	0.2927	1.259	0.993	1.054	0.643	1.148
1	0.3321	1.000	1.000	1.000	0.649	1.000
3	0.4850	0.4869	1.013	0.843	0.660	0.670
10	0.7281	0.2193	1.018	0.693	0.665	0.442
100	1.5719	0.0473	1.020	0.473	0.666	0.205
1000	3.3873	0.0102	1.020	0.323	0.667	0.095
$\mathrm{Pr} \to 0$	$0.5642\,\mathrm{Pr}^{1/2}$	$1.699\,\mathrm{Pr}^{-1/2}$	–	1.699	1/2	$0.742\,\mathrm{Pr}^{-1/2}$
$\mathrm{Pr} \to \infty$	$0.3387\,\mathrm{Pr}^{1/3}$	$1.020\,\mathrm{Pr}^{-2/3}$	1.020	–	2/3	$0.945\,\mathrm{Pr}^{-1/3}$

The local exponent rises from 1/2 to 2/3 and is already 0.649 at $\mathrm{Pr} = 1$, where $\mathcal{R} = 1$ exactly. This is why the Chilton–Colburn form $St\,\mathrm{Pr}^{2/3} = C_f/2$ is accurate to about $\pm 2\%$ for $0.5 \lesssim \mathrm{Pr} < \infty$ but overpredicts $\mathcal{R}$ by about 40% at $\mathrm{Pr} = 0.01$. A uniformly valid interpolation (Churchill and Ozoe), accurate to about 1.3%, is

$$Nu_x = \frac{0.3387\,Re_x^{1/2}\mathrm{Pr}^{1/3}}{\big[1 + (0.0468/\mathrm{Pr})^{2/3}\big]^{1/4}}. \quad (44)$$

Further laminar relations

Thermal-layer thickness at the end of a plate of length L.

	$\delta_{\theta,99}/L$	$\delta_q/L = 1/Nu_L$ (local)
$\mathrm{Pr} \ll 1$	$3.64\,Pe_L^{-1/2}$	$1.77\,Pe_L^{-1/2}$
$\mathrm{Pr} = 1$	$4.91\,Re_L^{-1/2}$	$3.01\,Re_L^{-1/2}$
$\mathrm{Pr} \gg 1$	$4.64\,Re_L^{-1/2}\mathrm{Pr}^{-1/3}$	$2.95\,Re_L^{-1/2}\mathrm{Pr}^{-1/3}$

The thin-layer assumption needs $\delta_\theta/L \ll 1$, i.e. $Pe_L \gg 1$.

Plate averages. Since τ_w and q''_w both scale as $x^{-1/2}$, $\overline{C_f} = 2C_{f,x}(L)$ and $\overline{Nu}_L = 2Nu_x(L)$, so the averaged analogy factor equals the local one.

Non-isothermal wall. For $\Theta_\mathrm{w} - \Theta_\infty = Cx^m$ and $\phi = (\Theta - \Theta_\infty)/(\Theta_\mathrm{w} - \Theta_\infty)$,

$$\phi'' + \frac{\mathrm{Pr}}{2} f\phi' - m\,\mathrm{Pr}\, f'\phi = 0, \qquad \phi(0) = 1,\ \phi(\infty) = 0, \qquad Nu_x = -\phi'(0) Re_x^{1/2}. \tag{45}$$

Uniform heat flux is $m = 1/2$. Numerically $Nu_x \to 0.4637\,Re_x^{1/2}\mathrm{Pr}^{1/3}$ for $\mathrm{Pr} \gg 1$. For $\mathrm{Pr} \ll 1$ the exact slug-flow solution $\phi = e^{-s^2} - \sqrt{\pi}\,s\,\mathrm{erfc}\,s$ gives $Nu_x = (\sqrt{\pi}/2)Pe_x^{1/2} = 0.886\,Pe_x^{1/2}$ and $\mathcal{R} \to 2.669\,\mathrm{Pr}^{-1/2}$.

Unheated starting length ξ. For $\mathrm{Pr} \ll 1$, $Nu_x = Nu_x|_{\xi=0}(1 - \xi/x)^{-1/2}$ at leading order. For $\mathrm{Pr} \gtrsim 0.6$ the integral method gives approximately $Nu_x = Nu_x|_{\xi=0}[1 - (\xi/x)^{3/4}]^{-1/3}$.

4 Turbulent boundary layer: what changes

Reynolds-averaged boundary-layer equations

Decompose $u = \bar{u} + u'$, $w = \bar{w} + w'$, $\Theta = \bar{\Theta} + \Theta'$. The averaged boundary-layer equations are

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{w}\frac{\partial \bar{u}}{\partial z} = \frac{1}{\rho}\frac{\partial \tau}{\partial z}, \qquad \frac{\tau}{\rho} = \nu\frac{\partial \bar{u}}{\partial z} - \overline{u'w'} \equiv (\nu + \nu_t)\frac{\partial \bar{u}}{\partial z}, \tag{46}$$

$$\bar{u}\frac{\partial \bar{\Theta}}{\partial x} + \bar{w}\frac{\partial \bar{\Theta}}{\partial z} = -\frac{1}{\rho c_p}\frac{\partial q}{\partial z}, \qquad \frac{q}{\rho c_p} = -\alpha\frac{\partial \bar{\Theta}}{\partial z} + \overline{w'\Theta'} \equiv -(\alpha + \alpha_t)\frac{\partial \bar{\Theta}}{\partial z}, \tag{47}$$

where $\tau(z)$ and $q(z)$ are the total (molecular plus turbulent) shear stress and heat flux, ν_t and α_t are eddy viscosity and eddy diffusivity, and $\mathrm{Pr}_t = \nu_t/\alpha_t$ is the turbulent Prandtl number. Setting $\nu_t = \alpha_t = 0$ recovers (14)–(15).

One exact formula for both regimes

Integrating $\partial\bar{u}/\partial z = (\tau/\rho)/(\nu + \nu_t)$ and $-\partial\bar{\Theta}/\partial z = (q/\rho c_p)/(\alpha + \alpha_t)$ across the layer gives

$$U_\infty = \int_0^\infty \frac{\tau/\rho}{\nu + \nu_t}\,dz, \qquad \Delta\Theta = \int_0^\infty \frac{q/(\rho c_p)}{\alpha + \alpha_t}\,dz. \tag{48}$$

Substituting into (3):

$$\boxed{\mathcal{R} = \frac{St_x}{C_{f,x}/2} = \frac{\displaystyle\int_0^\infty \frac{\tau/\tau_\mathrm{w}}{\nu + \nu_t}\,dz}{\displaystyle\int_0^\infty \frac{q/q''_\mathrm{w}}{\alpha + \alpha_t}\,dz}} \tag{49}$$

This is exact within the boundary-layer approximation (the eddy diffusivities are definitions). It shows clearly what each regime depends on.

- **Laminar** ($\nu_t = \alpha_t = 0$): equation (49) reduces to $\mathcal{R} = \mathrm{Pr}^{-1}\,\delta_\tau/\delta_q$ with $\delta_\tau = \int_0^\infty (\tau/\tau_\mathrm{w})\,dz$ and $\delta_q = \int_0^\infty (q/q''_\mathrm{w})\,dz$, which is identity (17). The diffusivities are constants; the whole answer lies in *how fast the fluxes decay* away from the wall. By (46)–(47) that decay rate is the convective (inertia) term: $\partial\tau/\partial z = \rho(\bar{u}\partial_x\bar{u} + \bar{w}\partial_z\bar{u})$. This is the inertia–diffusion balance of Section 2.

- **Turbulent**: near the wall the fluxes are almost constant and the answer lies in *how the diffusivities vary with* z. This is explained next.

The two-layer structure: where inertia matters and where it does not

Inner layer: inertia is negligible. At the wall $\bar{u} = \bar{w} = 0$ and $\partial\tau/\partial z|_\mathrm{w} = dp/dx = 0$. Expanding near the wall, the convective terms are $O(z^2)$, so for $z \ll \delta$

$$\tau(z) \approx \tau_\mathrm{w}, \qquad q(z) \approx q''_\mathrm{w} \qquad \text{(constant-flux layer)}. \tag{50}$$

In wall units, $u_\tau = \sqrt{\tau_\mathrm{w}/\rho}$, $z^+ = zu_\tau/\nu$, $U^+ = \bar{u}/u_\tau$, $\Theta^+ = (\Theta_\mathrm{w} - \bar{\Theta})\rho c_p u_\tau/q''_\mathrm{w}$ and $\nu_t^+ = \nu_t/\nu$, equations (46)–(47) become

$$\frac{dU^+}{dz^+} = \frac{1}{1+\nu_t^+}, \qquad \frac{d\Theta^+}{dz^+} = \frac{1}{\mathrm{Pr}^{-1} + \nu_t^+/\mathrm{Pr}_t}. \tag{51}$$

The wall fluxes are set by the *resistance* of this near-wall layer (molecular sublayer in series with turbulent transport), not by a convection–diffusion balance.

Outer layer: inertia balances turbulent stress, not viscosity. In the outer part the velocity defect $U_\infty - \bar{u}$ scales with u_τ, and $\partial\tau/\partial z \sim \tau_\mathrm{w}/\delta$. The momentum balance is

$$\boxed{U_\infty \frac{u_\tau}{x} \sim \frac{u_\tau^2}{\delta}} \quad \Longrightarrow \quad \frac{\delta}{x} \sim \frac{u_\tau}{U_\infty} = \sqrt{\frac{C_{f,x}}{2}}. \tag{52}$$

Compare the laminar balance $U_\infty^2/x \sim \nu U_\infty/\delta^2$: the molecular viscosity has disappeared from the outer balance. Viscosity enters only through the sublayer thickness ν/u_τ. Matching the inner and outer layers through the log law, $U_\infty/u_\tau = \kappa^{-1}\ln(\delta u_\tau/\nu) + B$, gives an implicit friction law. Its constants $\kappa \approx 0.41$, $B \approx 5$ come from experiments, and commonly used fits such as $C_{f,x}/2 \approx 0.0296\,Re_x^{-1/5}$ are empirical.

The turbulent analogy factor in terms of equations

Equations (49) and (51) give the exact relation

$$St_x = \frac{u_\tau}{U_\infty\,\Theta_e^+}, \qquad \frac{C_{f,x}}{2} = \frac{1}{(U_e^+)^2} \quad \Longrightarrow \quad \boxed{\mathcal{R} = \frac{U_e^+}{\Theta_e^+}} \tag{53}$$

where the subscript e denotes the boundary-layer edge. If $\mathrm{Pr} = \mathrm{Pr}_t = 1$ and τ/τ_w, q/q''_w have the same profile, then $\Theta^+ \equiv U^+$ and $\mathcal{R} = 1$: the original (turbulent) Reynolds analogy.

In general, integrating (51) across the sublayer and the turbulent region (where $\nu_t \gg \nu$ and $\alpha_t \gg \alpha$) gives $\Theta_e^+ = \mathrm{Pr}_t\,U_e^+ + P(\mathrm{Pr})$, where P is the extra thermal resistance of the sublayer. Hence

$$\boxed{\mathcal{R} = \frac{1}{\mathrm{Pr}_t + P(\mathrm{Pr})\sqrt{C_{f,x}/2}}} \tag{54}$$

Two classical models follow:

- **Prandtl–Taylor** (laminar sublayer to $z^+ = 5$, then $\mathrm{Pr}_t = 1$): $P = 5(\mathrm{Pr} - 1)$, so $\mathcal{R} = \left[1 + 5\sqrt{C_{f,x}/2}\,(\mathrm{Pr} - 1)\right]^{-1}$.
- **von Kármán** (adds a buffer layer $5 < z^+ < 30$): $\mathcal{R} = \left\{1 + 5\sqrt{C_{f,x}/2}\,\left[\mathrm{Pr} - 1 + \ln\!\left(1 + \tfrac{5}{6}(\mathrm{Pr} - 1)\right)\right]\right\}^{-1}$.

The key difference from the laminar case: the turbulent $\mathcal{R}$ depends on Re_x (through $C_{f,x}$) and on Pr_t, not on Pr alone.

Large Pr. Near the wall, no-slip and continuity give $u' \propto z$, $w' \propto z^2$, $\Theta' \propto z$, so $\alpha_t^+ = Az^{+3}$ (if $\overline{w'\Theta'}$ keeps its leading Taylor term). For $\mathrm{Pr} \gg 1$ the whole thermal resistance lies where α_t is comparable with α, i.e. $z^+ \sim (A\,\mathrm{Pr})^{-1/3}$, deep inside the viscous sublayer:

$$\Theta_e^+ \approx \int_0^\infty \frac{dz^+}{\mathrm{Pr}^{-1} + Az^{+3}} = \frac{2\pi}{3\sqrt{3}} A^{-1/3}\,\mathrm{Pr}^{2/3} \quad \Longrightarrow \quad \mathcal{R} \approx \frac{0.827\,A^{1/3}}{\sqrt{C_{f,x}/2}}\,\mathrm{Pr}^{-2/3}. \tag{55}$$

So $\mathrm{Pr}^{-2/3}$ reappears, but for a different reason: in the laminar case it comes from convection by a linear velocity profile (Lévêque); in the turbulent case from an eddy diffusivity growing as z^3. If instead $\alpha_t^+ \propto z^{+4}$, the same integral gives $\mathrm{Pr}^{-3/4}$. The exponent therefore depends on the wall-limiting behaviour of turbulence, and the constant A is not given by theory.

Small Pr. Molecular conduction $\alpha^+ = 1/\mathrm{Pr}$ dominates $\alpha_t^+ \approx \kappa z^+/\mathrm{Pr}_t$ up to $z^+ \sim \mathrm{Pr}_t/(\kappa\,\mathrm{Pr}) \gg 1$, far beyond the viscous sublayer. If this exceeds $\delta^+ = \delta u_\tau/\nu$, the thermal field is conduction-dominated across the whole layer and behaves like the laminar slug-flow problem. Otherwise $\mathcal{R}$ depends on Pr_t in the log and outer regions, and Pr_t itself rises well above 0.85–0.9 when Pr is small. Models such as (54) that assume $\mathrm{Pr}_t \approx 1$ fail here.

How much the models disagree. At $Re_x = 10^6$ ($C_{f,x}/2 \approx 0.0296\,Re_x^{-1/5} = 0.00187$):

Pr	Chilton–Colburn $\mathrm{Pr}^{-2/3}$	Prandtl–Taylor	von Kármán
0.01	21.5	1.27	2.44
0.7	1.27	1.07	1.15
1	1.00	1.00	1.00
7	0.273	0.435	0.373
100	0.046	0.045	0.043

At $\mathrm{Pr} = 7$ the predictions differ by up to 60%, and at $\mathrm{Pr} = 0.01$ by an order of magnitude. This spread is the gap that only DNS or experiment can close.

Laminar versus turbulent: summary

	Laminar	Turbulent
Dominant balance	inertia $\sim$ molecular diffusion, across the whole layer	outer: inertia $\sim$ turbulent stress; inner: constant flux, inertia negligible
Layer growth	$U_\infty^2/x \sim \nu U_\infty/\delta^2$, $\delta/x \sim Re_x^{-1/2}$	$U_\infty u_\tau/x \sim u_\tau^2/\delta$, $\delta/x \sim u_\tau/U_\infty$
What sets the wall flux	flux-decay thickness (δ_τ, δ_q)	near-wall resistance, profile of ν_t, α_t
$\mathcal{R}$ depends on	Pr only	Pr, Pr_t, Re_x
$\mathrm{Pr} \gg 1$	$1.020\,\mathrm{Pr}^{-2/3}$ (exact)	$\propto (C_f/2)^{-1/2}\mathrm{Pr}^{-2/3}$ if $\alpha_t \propto z^3$ (model)
$\mathrm{Pr} \ll 1$	$1.699\,\mathrm{Pr}^{-1/2}$ (exact)	no closed form; needs Pr_t from DNS

5 Limits of the theory: where DNS or experiment are needed

Maximum wall values. In the laminar similarity solution, τ_w and q''_w both scale as $x^{-1/2}$. Both are singular at $x = 0$ and their ratio is constant, so $q''_{\mathrm{w,max}}/\tau_{\mathrm{w,max}}$ is not defined by boundary-layer theory. It is set by the finite leading-edge region, plate thickness or bluntness, or rarefaction.

Open questions.

1. **Leading-edge region.** For $Re_x \lesssim O(10^2$–$10^3)$ the boundary-layer equations are invalid. The location and value of $\tau_{\mathrm{w,max}}$ and $q''_{\mathrm{w,max}}$ need full Navier–Stokes computations or measurement.

2. **Axial conduction at low Pr.** When $Pe_x = O(1)$ while $Re_x \gg 1$, streamwise conduction matters and heat conducts upstream. The elliptic energy equation must be solved numerically.

3. **Real liquid metals (**$\mathrm{Pr} \sim 0.005$–0.03**).** The $\mathrm{Pr}^{-1/2}$ asymptote converges slowly; the full similarity solution is needed. Experiments are limited by wetting, oxide layers and contact resistance.

4. **Turbulent closure.** The profile of $\mathrm{Pr}_t(z^+, \mathrm{Pr})$, the near-wall constant A and exponent of α_t, the sublayer resistance $P(\mathrm{Pr})$, and the friction-law constants are not given by theory. High-Schmidt-number mass-transfer experiments find exponents near 0.70, between 2/3 and 3/4.

5. **Transition.** Location, intermittency, and the analogy factor through transition all require data.

6. **Pressure gradient and separation.** Falkner–Skan wedge flows are solvable, but $dp/dx \neq 0$ breaks the analogy. Near separation $C_f \to 0$ while St stays finite, so $\mathcal{R} \to \infty$.

7. **Property variation, compressibility, buoyancy.** The available corrections (property-ratio or reference-temperature methods, recovery factor, Richardson-number limits for mixed convection) are approximate.

8. **Free-stream turbulence, roughness, three-dimensionality, and conjugate wall conduction.** None of these is captured by the theory above.